Harmonic functions. Poisson's formula. Schwarz's theorem

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Def. Let $u \in C^2(\Omega)$ (twice continuously real differentiable)
u is called harmonic if $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mathrm{div}(\nabla u) = 0$.

Notation. $\mathrm{Harm}(\Omega)$. We'll consider real-valued harmonic functions.

As for holomorphic: $u \in \mathrm{Harm}(S)$ means $\exists O \supset S$-open, $u \in \mathrm{Harm}(O)$

Reminder. $f \in \mathcal{A}(\Omega) \Rightarrow \mathrm{Re} f, \mathrm{Im} f \in \mathrm{Harm}(\Omega)$.
Follows from Cauchy-Riemann.

Is the opposite true?
Not always: $\log|z| \in \mathrm{Harm}(\mathbb{C}\setminus 0)$ ($\forall z$ $\log|z| = \mathrm{Re}\log z$ locally, i.e. in $B(z,|z|)$ there is a branch of logarithm).

But if $\exists f \in \mathcal{A}(\mathbb{C}\setminus 0)$: $\mathrm{Re} f = u$
then $f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{1}{z}$, and $\oint_{\{|z|=1\}} \frac{dz}{z} \neq 0 \Rightarrow \frac{1}{z}$ has no antiderivative! Contradiction!
Cauchy-Riemann

But true in simply-connected regions:

Theorem. Let Ω be a simply-connected region.
Let $u \in \mathrm{Harm}(\Omega)$. Then $\exists f \in \mathcal{A}(\Omega)$: $u = \mathrm{Re} f$
If $u = \mathrm{Re} f_1 = \mathrm{Re} f_2$ then $f_1 - f_2 \equiv \mathrm{const} \in i\mathbb{R}$.

Corollary $u \in \mathrm{Harm}(\Omega)$ $(\forall \Omega) \Rightarrow u \in C^\infty(\Omega)$.

Proof (Theorem $\Rightarrow$ Corollary).
Let $z \in \Omega$, $\exists B(z,r) \subset \Omega$. $B(z,r)$ – simply connected.
So $\exists f \in \mathcal{A}(B(z,r))$: $u = \mathrm{Re} f$. $f \in C^\infty \Rightarrow u \in C^\infty$ ∎

Proof (of Theorem)
Let $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = u_1 + i v_1$.

$$\left.\begin{array}{l} \frac{\partial u_1}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial v_1}{\partial y} \\ \frac{\partial u_1}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial v_1}{\partial x} \end{array}\right\} \Rightarrow g \in \mathcal{A}(\Omega).$$

Ω – simply connected. So $\exists f \in \mathcal{A}(\Omega)$: $g(z) = f'(z)$.

Another way: define $f(z) = u(z_0) + \oint_{\gamma_{z_0,z}} g(\zeta)\,d\zeta$, $\gamma_{z_0,z}$ – any path from z_0 to z.

Let $f(z) = U(z) + iV(z)$.

$$g(z) = \frac{\partial U}{\partial x} - i\frac{\partial V}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \Rightarrow \frac{\partial(U-u)}{\partial x} = \frac{\partial(U-u)}{\partial y} = 0 \Rightarrow U = u + \mathrm{const}.$$

So $\mathrm{Re}(f - \mathrm{const}) = u$.
If $\mathrm{Re} f_1 = \mathrm{Re} f_2 \Leftrightarrow \mathrm{Re}(f_1 - f_2) \equiv 0 \Rightarrow f_1 = f_2 + ic$. ∎

Theorem (Mean Value Property).
Let $u \in \mathrm{Harm}(\Omega)$, $\overline{B(z_0,r)} \subset \Omega$.
Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt = \frac{1}{2\pi r}\int_{C_r} u(z)\,|dz|, \quad C_r = \{|z - z_0| = r\}$$

Proof. Take $r' > r$: $B(z_0,r') \subset \Omega$.
$u \in \mathrm{Harm}(B(z_0,r')) \Rightarrow \exists f \in \mathcal{A}(B(z_0,r'))$, $u = \mathrm{Re} f$.

By Cauchy:

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} \cdot ire^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

Take Re of both sides ∎

Another proof (and stronger statement):
Let $\Omega = \{r_1 < |z - z_0| < r_2\}$, $u \in \mathrm{Harm}(\Omega)$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt = \alpha \log r + \beta, \quad r_1 < r < r_2, \quad \alpha, \beta \in \mathbb{R}$$

In particular, if $u \in \mathrm{Harm}(B(z_0,R))$, $\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt = \beta = u(z_0)$

In $\mathbb{R}^d$, $d \geq 3$: $\Delta r^{d-2} = 0$

$$\int_{S_r} u(\zeta)\,d\sigma(\zeta) = \frac{\alpha}{r^{d-2}} + \beta$$

When $d = 2$:
$r^{d-2} = \log\frac{1}{r}$.
$d = 2$: $\Delta \log r = 0$

Proof. Let $z_0 = 0$ (can shift everything)

Define $v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(ze^{it})\,dt$, $z \in \Omega$ – average over circle $\{w : |w| = |z|\}$

Then $\forall \theta \in \mathbb{R}$: $v(ze^{i\theta}) = v(z)$.

So $v(z) = v(|z|)$.

Also $\Delta v(z) = \frac{1}{2\pi}\int_0^{2\pi} \Delta u(ze^{it})\,dt = 0$

In polar coordinates, $\Delta u(re^{i\theta}) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

In particular, since $v(re^{i\theta}) = v(r)$, we have

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 0 \iff \frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0 \iff r\frac{\partial v}{\partial r} = \alpha \iff v(r) = \alpha \log r + \beta$$

Siméon Poisson

The Poisson Formula

Let $u \in Harm(\overline{\mathbb{D}})$. $u = Re f$. $(f \in \mathcal{A}(\overline{\mathbb{D}}))$.

Let $z_0 \in \mathbb{D}$. $S(z) := \frac{z+z_0}{1+\overline{z}_0 z}$. Then

$u \circ S = Re(f \circ S)$. $f \circ S \in \mathcal{A}(\overline{\mathbb{D}}) \Rightarrow u \circ S \in Harm(\overline{\mathbb{D}})$

By Mean Value Property:

$$u(z_0) = u(S(0)) = \frac{1}{2\pi}\int_0^{2\pi} u(S(e^{it}))\,dt$$

Observe: $S(e^{it}) = \frac{e^{it}+z_0}{1+\overline{z}_0 e^{it}} =: e^{i\theta}$

So $e^{it} = \frac{e^{i\theta}-z_0}{1-\overline{z}_0 e^{i\theta}}$ $\qquad w = z_0$

$$dt = \frac{de^{it}}{ie^{it}} = \frac{ie^{i\theta} - |w|^2 ie^{i\theta}}{(1-\overline{w}e^{i\theta})^2}\cdot\left(-i\,\frac{1-\overline{w}e^{i\theta}}{e^{i\theta}-w}\right)d\theta$$

$$\frac{1-|w|^2}{|e^{i\theta}-w|^2}\,d\theta.$$

So we have:

Poisson formula: $$u(w) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|w|^2}{|e^{i\theta}-w|^2}\,u(e^{i\theta})\,d\theta$$

Poisson formula for $B(z_0,r)$

If $u \in Harm(\overline{B(z_0,r)})$ then $\forall\, w \in B(z_0,r)$

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|w-z_0|^2}{|re^{i\theta}+z_0-w|^2}\,u(z_0+re^{i\theta})\,d\theta.$$

Obtained by rescaling.

Remark. As in homework, can assume less:

$u \in Harm(B(z_0,r))$, $u \in C(\overline{B(z_0,r)})$.

Schwarz Theorem

Let U be piecewise continuous on $|z|=1$.

Def Poisson integral of U:

$$P_U(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} U(e^{i\theta})\,d\theta, \quad |z|<1.$$

Remark $U \mapsto P_U$ is a linear functional:

$$P_{\lambda U+\mu V} = \lambda P_U + \mu P_V.$$

Also, if $U\equiv 1$, then $P_U \equiv 1$. (Poisson formula for $u\equiv 1$!).
if $U \le V$, then $P_U \le P_V$ (since $\frac{1-|z|^2}{|z-e^{i\theta}|^2}>0$).

Theorem. $P_U(z)$ is harmonic for $z\in\mathbb{D}$.
If U is continuous at $e^{i\theta_0}$, then

$$\lim_{\substack{z\to e^{i\theta_0}\\ |z|<1}} P_U(z) = U(e^{i\theta_0})$$

Proof. Observe that

$$\frac{1-|z|^2}{|z-e^{i\theta}|^2} = \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) \qquad d\theta = \frac{d\zeta}{i\zeta}$$

So $P_U(z) = \frac{1}{2\pi}\mathrm{Re}\left(\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} U(e^{i\theta})\,d\theta\right) = \frac{1}{2\pi}\mathrm{Re}\oint_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\frac{U(\zeta)}{i\zeta}\,d\zeta$

Observe that

$f(z) := \oint_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\frac{U(\zeta)}{i\zeta}\,d\zeta$ is analytic in z, since $\frac{\zeta+z}{\zeta-z} = 1+2\sum_{k=1}^{\infty}\left(\frac{z}{\zeta}\right)^k$ uniformly in ζ (fixed z)

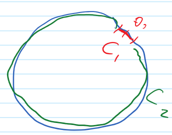
So $f(z) = \underbrace{\oint_{\mathbb{T}} \frac{U(\zeta)}{i\zeta}\,d\zeta}_{a_0} + \sum_{k=1}^{\infty} z^k \underbrace{\left(2\oint_{\mathbb{T}}\frac{U(\zeta)}{i\zeta^{k+1}}\,d\zeta\right)}_{a_k} = \sum_{k=0}^{\infty} a_k z^k.$

So $P_U(z)$ is harmonic.

Let U be continuous in $e^{i\theta_0}$. Fix $\varepsilon>0$, choose $\delta>0$: $|\theta-\theta_0|<\delta \Rightarrow |U(e^{i\theta})-U(e^{i\theta_0})|<\varepsilon$.

Let $C_1 = \{e^{i\theta} : |\theta-\theta_0|_{\mathrm{mod}\,2\pi} < \delta\}$, $C_2 = \{e^{i\theta} : |\theta-\theta_0|_{\mathrm{mod}\,2\pi} \ge \delta\}$.

Then $P_U(z) = \oint_{C_1} + \oint_{C_2}$.



Observe that as $z\to e^{i\theta_0}$, $\frac{1-|z|^2}{|z-e^{i\theta}|^2}\to 0$ uniformly on C_2.

Indeed, $|e^{i\theta_0}-z|<\delta' \ll \delta \Rightarrow |z-e^{i\theta}| \ge \delta-\delta'$, $1-|z|^2 = (1-|z|)(1+|z|) < 2\delta'$,

so $\frac{1-|z|^2}{|z-e^{i\theta}|^2} \le \frac{\delta'}{(\delta-\delta')^2} \xrightarrow[\delta'\to 0]{} 0$.

So, as $z\to e^{i\theta_0}$, $\oint_{C_2} \frac{1-|z|^2}{|z-\zeta|^2}\frac{U(\zeta)}{i\zeta}\,d\zeta \to 0$.

On the other hand: $P_1 \equiv 1$

$\frac{1}{2\pi}\oint_{C_1} \frac{1-|z|^2}{|z-\zeta|^2}\frac{U(\zeta)}{i\zeta}\,d\zeta - U(e^{i\theta_0}) = \frac{1}{2\pi}\oint_{C_1}\frac{1-|z|^2}{|z-\zeta|^2}\frac{U(\zeta)}{i\zeta}\,d\zeta -$

$\frac{1}{2\pi}\oint_{\mathbb{T}}\frac{U(e^{i\theta_0})}{i\zeta}\frac{1-|z|^2}{|z-\zeta|^2}\,d\zeta =$

$\underbrace{\frac{1}{2\pi}\oint_{C_1}\left(-U(e^{i\theta_0})+U(\zeta)\right)\frac{1-|z|^2}{|z-\zeta|^2}\frac{d\zeta}{i\zeta}}_{\mathrm{I}} + \underbrace{\frac{1}{2\pi}\oint_{C_2} U(e^{i\theta_0})\frac{1-|z|^2}{|z-\zeta|^2}\,d\zeta}_{\mathrm{II}}.$

$|\mathrm{I}| \le \frac{1}{2\pi}\varepsilon\oint_{\mathbb{T}}\frac{1-|z|^2}{|z-\zeta|^2}\frac{d\zeta}{\zeta} = \varepsilon.$

$|U(e^{i\theta_0})-U(\zeta)|<\varepsilon$

On C_2, $\frac{1-|z|^2}{|z-\zeta|^2}\to 0$ uniformly as $z\to e^{i\theta_0}$.

so $\mathrm{II}\to 0$ as $|z|\to 0$.

So if $|e^{i\theta_0}-z|$ is small, $|P_U(z)-U(e^{i\theta_0})|<2\varepsilon$

Approximate identity:

$\varphi_\varepsilon(x)$:

1) $\varphi_\varepsilon(x)\ge 0$
2) $\int\varphi_\varepsilon(x)\,dx = 1$
3) $\lim_{\varepsilon\to 0}\int_{|x-x_0|>\delta}\varphi_\varepsilon(x)\,dx = 0 \;\forall\delta>0.$

Then $\forall f$ continuous at x_0:

$\int f(x)\varphi_\varepsilon(x)\,dx \to f(x_0).$

$C = \int C\varphi_\varepsilon(x)\,dx$

$\int f(x)\varphi_\varepsilon(x)\,dx - C =$

$\int (f(x)-C)\varphi_\varepsilon(x)\,dx$

So $\text{II} \to 0$ as $|z| \to 0$.

So if $|e^{i\theta_0} - z|$ is small, $|P_V(z) - V(e^{i\theta_0})| < 2\varepsilon$

Then $\forall f$ continuous at x_0:

$\int f(x)\varphi_\varepsilon(x)\,dx \to f(x_0)$.

$\int (f(x) - c)\,\varphi_\varepsilon(x)\,dx$

In Physics terms

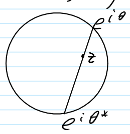
$\langle f, \varphi_\varepsilon \rangle \to \langle f, \delta \rangle$

Remark The proof gives us a formula for $f \in A(\Omega)$ such that $\operatorname{Re} f = P_V$:

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\zeta + z}{\zeta - z} V(\zeta) \frac{d\zeta}{\zeta} + iC. \text{ (Schwarz formula).}$$

If $u \in \operatorname{Harm}(\overline{\mathbb{D}})$, $V(\zeta) = u(\zeta)$.

Schwarz's geometric interpretation.



For $e^{i\theta} \in \mathbb{T}$, $z \in \mathbb{D}$, let θ^* is such that $e^{i\theta}, z, e^{i\theta^*}$ form a line.

By direct computation.

$$1 - |z|^2 = (z - e^{i\theta})(e^{-i\theta^*} - \bar{z})$$

Indeed. $e^{i\theta^*} = z + t(z - e^{i\theta})$ for some $t > 0$,

Plug in $t_0 = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}$, then $|z + t_0(z - e^{i\theta})| = \left| z + \frac{1 - |z|^2}{|z - e^{i\theta}|^2}(z - e^{i\theta}) \right| =$

$\left| z + \frac{1 - |z|^2}{\bar{z} - e^{-i\theta}} \right| = 1$. So $e^{i\theta^*} = z + \frac{1 - |z|^2}{(\bar{z} - e^{-i\theta})}$.

$d e^{i\theta} = i e^{i\theta} d\theta$ $\quad$ $d e^{i\theta^*} = i e^{i\theta^*} d\theta^*$

So $\frac{d\theta^*}{d\theta} = \left| \frac{e^{i\theta^*} - z}{e^{i\theta} - z} \right| = \frac{1 - |z|^2}{|e^{i\theta^*} - z|^2}$ $\left(\text{Since } |e^{i\theta^*} - z| = |e^{-i\theta^*} - \bar{z}| = \frac{1 - |z|^2}{|z - e^{i\theta}|} \right)$.

So $P_V(z) = \frac{1}{2\pi} \int_0^{2\pi} V(\theta)\, d\theta^* = \frac{1}{2\pi} \int_0^{2\pi} V(\theta^*)\, d\theta$.

Schwarz theorem in $B(z_0, r)$:

V is piecewise continuous on $\{|z - z_0| = r\}$. Then

$u(z) := \frac{1}{2\pi} \int \frac{r^2 - |z - z_0|^2}{|z - z_0 - re^{it}|^2} V(re^{it} + z_0)\, dt$ is harmonic in $B(z_0, r)$,

$\lim_{z \to z_0 + re^{it}} u(z) = V(z_0 + re^{it})$ if V is continuous at $z_0 + re^{it}$.

Peter Gustav Lejeune Dirichlet

Dirichlet problem:

Given $f \in C(\partial\Omega)$, find u: 1) $u \in C(\overline{\Omega})$

2) $u \in \operatorname{Harm}(\Omega)$ $\quad$ $\min \iint_\Omega |\nabla u|^2\, dx\, dy$.

3) $u|_{\partial\Omega} = f$.

We solved it for $\Omega = B(z_0, r)$.